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ABSTRACT

Elliptic boundary value problems of the form $Lu = g(x, u)$ in Ω (omega) and $u = 0$ on the boundary of Ω (omega) are studied where g is singular in that $g(x, r) \to \infty$ (goes to infinity) uniformly as $r \to 0+$ (goes to zero from above). Existence of classical and generalized solutions is established and an associated nonlinear eigenvalue problem is treated. A detailed study is made of the behaviour of the solutions and their gradients near the boundary of Ω (omega). This leads to global estimates for the modulus of continuity of solutions.

AMS(MOS) Subject Classification - 35J65

Key Words - Regularity, singular Dirichlet problem, nonlinear eigenvalue problem.

Work Unit Number 1 - Applied Analysis

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ON A DIRICHLET PROBLEM WITH A SINGULAR NONLINEARITY

M. G. Crandall, P. H. Rabinowitz and L. Tartar

This paper concerns nonlinear elliptic boundary value problems of the form

$$(0.1)\quad \begin{cases} Lu = -\sum_{i,j=1} a_{ij}(x) \dfrac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \dfrac{\partial u}{\partial x_i} + c(x)\, u = g(x,u),\ x \in \Omega \\ \\ u(x) = 0, \quad x \in \partial\Omega \ , \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $\partial\Omega$ is the boundary of Ω, the coefficients of L are real, $c \geq 0$ and $\sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j > 0$ for $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. The principle feature of interest here is that we assume g is singular in the sense that $g(x,r)$ is only defined for $r > 0$ and $g(x,r) \to +\infty$ as $r \to 0+$ uniformly for $x \in \overline{\Omega}$. Obviously (0.1) cannot then have a solution $u \in C^2(\overline{\Omega})$ and, in the cases we discuss, there may be no solutions of class $C^1(\overline{\Omega})$ or $W_0^{1,2}(\Omega)$. However, under appropriate assumptions, we will obtain a classical solution of (0.1), i.e. a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ with $u > 0$ in Ω. In particular, if $g, L, \partial\Omega$ are sufficiently smooth and $g(x,r)$ is bounded from above uniformly for $x \in \overline{\Omega}$ and $r \geq 1$, then (0.1) has a classical solution (Corollary 1.10). If the coefficients of L and g are merely continuous, solutions of (0.1) still exist in a generalized sense made precise later (Theorem 1.21).

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A second object of study here is the behaviour of solutions of (0.1) near $\partial\Omega$ when g is singular. As a consequence of this study, stronger global (i.e. in $\overline{\Omega}$) regularity properties than continuity of solutions are obtained. For example, if $g(x,r) = r^{-\alpha}$, $\alpha > 1$, then we show solutions u of (0.1) lie in the Hölder class $C^{2/(\alpha+1)}(\overline{\Omega})$.

The existence of solutions of (0.1) is discussed in Section 1. The special case in which the map $r \to g(x,r)$ is nonincreasing admits an especially simple solution and is studied first. Then more general cases are treated by means of the nonlinear eigenvalue problem

(0.2) $$Lu = \lambda g(x,u), \quad x \in \Omega; \; u = 0, \quad x \in \partial\Omega .$$

The boundary behaviour and regularity are discussed in Section 2. First the rate at which $u(x) \to 0$ when $x \to \partial\Omega$ is determined. This is used to study the behaviour of $|\operatorname{grad} u(x)|$ as $x \to \partial\Omega$. Lastly as a consequence of these results, we obtain an estimate for the modulus of continuity of u in $\overline{\Omega}$.

There seems to have been little work done on singular problems such as (0.1) in the literature. After our main results were obtained we learned of the work of Fulks and Maybee [5] and of Stuart [10]. In [5], the authors treat the existence question for the equation

(0.3) $$u_t - \Delta u = g(x,t,u), \quad x \in \Omega \subset \mathbb{R}^n, \quad t > 0$$

coupled with initial and boundary conditions for a class of functions g which are nonincreasing in u. Assuming that $g(x,t,r) \to g(x,r)$ as $t \to \infty$ they also obtain classical solutions of the corresponding elliptic boundary value problem upon letting $t \to \infty$. Their proofs involve, in part, an interesting argument reminiscent of the Schwarz alternating procedure for Laplace's equation. In [10] Stuart studies the existence of classical solutions to (0.1) under hypotheses related to those of Corollary 1.10 below. Stuart's attack is based on finding sub- and supersolutions for approximate problems together with appropriate a priori estimates.

Singular problems somewhat related to (0.1) have also been treated in the context of integral equations. Nowosad [7] studied existence of solutions of the equation

$$u(x) = \int_0^1 K(x,y)\,(u(y))^{-1}\,dy \tag{0.4}$$

where K is a positive semidefinite kernel satisfying $\int_0^1 K(x,y)dy \geq \delta > 0$. This work was generalized by Karlin and Nirenberg [6] who treated

$$u(x) = \int_0^1 K(x,y)(u(y))^{-\alpha}\,dy \tag{0.5}$$

where $\alpha > 0$, $K \geq 0$, K is continuous and $K(x,x) > 0$ for $x \in [0,1]$. These authors also obtained some results concerning the nonlinear eigenvalue problem

$$\lambda u(x) = \int_0^1 K(x,y)\,\varphi(y,u(y))dy\ . \tag{0.6}$$

Another abstract result in the same direction was given by Ramalho [9]. More recently Stuart [11] studied existence for

$$(0.7) \qquad u(x) = g(x) + \int_{\Omega} K(x,y)\, f(y, u(y))dy$$

where singular functions were permitted.

The results of [6], [7], [8], [10] have the common feature that although the nonlinearity treated is singular at $u = 0$, the solution obtained is strictly positive in the (closed) domain of definition. Therefore the singularity plays a minor role in comparison with the results of [5], [9] and the current work.

The authors are indebted to Louis Nirenberg for a suggestion which led us to Theorem 2.5.

Section 1.

In this section the existence of classical solutions of (0.1) will be established when $\partial\Omega$, g and the coefficients of L are sufficiently smooth. Then the existence of generalized solutions of (0.1) when g and the coefficients of L are merely continuous will be proved. For simplicity, we take "sufficiently smooth" to mean that the coefficients of L are in $C^1(\bar{\Omega})$, $\partial\Omega$ is of class C^3 and $g \in C^1(\bar{\Omega} \times (0,\infty))$. This allows use of the Schauder theory with any exponent $\alpha \in (0,1)$ in the course of discussion. (Alternatively, $\alpha \in (0,1)$ could be fixed and corresponding modifications made in the assumptions and proofs below.) Sufficient smoothness is assumed through equation (1.19) below. Concerning g we will further assume

$$(g_1) \qquad \lim_{r \to 0+} g(x,r) = \infty \text{ uniformly for } x \in \bar{\Omega}\,.$$

We begin with the important special case in which also

(g_2) $\qquad g(x,r)$ is nonincreasing in $r \in (0,\infty)$ for $x \in \bar{\Omega}$.

The solvability of (0.1) under (g_1)-(g_2) is covered by the results of [5] (when $L = -\Delta$) and [10]. However, a simpler and more direct proof is presented here.

Theorem 1.1. If g satisfies (g_1)-(g_2) then (0.1) possesses a unique classical solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ with $u > 0$ in Ω .

Proof of Theorem 1.1. The existence will be established by solving the approximate problems

$$(1.2) \qquad \begin{cases} Lu_\varepsilon = g(x, \varepsilon + u_\varepsilon) & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

for $\varepsilon > 0$ and then showing the convergence of u_ε as $\varepsilon \to 0+$ to a solution u . The next two lemmas provide the basic information needed to carry out this process. In what follows $C^{j,\alpha}(\bar{\Omega})$ denotes the set of j-times continuously differentiable functions on $\bar{\Omega}$ whose derivatives of order j are Hölder continuous with exponent α and $C^{0,\alpha}(\bar{\Omega}) = C^\alpha(\bar{\Omega})$.

Lemma 1.3. Let (g_1) and (g_2) hold and $\varepsilon_0 > 0$ be such that $g(x,\varepsilon) > 0$ for $x \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$. If $0 < \varepsilon \leq \varepsilon_0$, then

(i) (1.2) has a unique nonnegative classical solution u_ε .

(ii) $u_\varepsilon \in C^{2,\alpha}(\bar{\Omega})$ for any $\alpha \in (0,1)$,

(iii) $u_\varepsilon(x) > 0$ for $x \in \Omega$ and

(1.4) $$u_\varepsilon \geq u_\delta, \quad \varepsilon + u_\varepsilon \leq \delta + u_\delta \quad \text{for } 0 < \varepsilon \leq \delta \leq \varepsilon_0 .$$

<u>Proof of Lemma 1. 3.</u> The existence of u_ε will follow once we exhibit an ordered pair of sub- and supersolutions of (1. 2) (see, e.g., [2]). Since $0 < \varepsilon \leq \varepsilon_0$, $L0 = 0 < g(x, \varepsilon) = g(x, \varepsilon+0)$ and therefore 0 is a subsolution of (1. 2). Next define $w \in C^{2,\alpha}(\bar{\Omega})$ by $Lw = g(x, \varepsilon)$ in Ω and $w = 0$ on $\partial\Omega$. The Schauder theory for linear elliptic equations ([1]) assures the existence and uniqueness of w. By the maximum principle $w > 0$ in Ω so, by (g_2), $Lw = g(x, \varepsilon) \geq g(x, \varepsilon+w)$ in Ω. Thus w is a supersolution of (1. 2) and $[0, w]$ is the desired ordered pair. Since $\frac{\partial}{\partial r} g(x, \varepsilon+r)$ is bounded for $0 \leq r \leq w(x)$, $x \in \bar{\Omega}$, a theorem of Amann [2] implies that (1. 2) has a solution $u_\varepsilon \in C^{2,\alpha}(\bar{\Omega})$ with $0 < u_\varepsilon \leq w$ in Ω. The uniqueness of u_ε follows once we establish (1. 4) for arbitrary solutions u_ε, u_δ of the ε and δ problems. Let $0 < \varepsilon \leq \delta \leq \varepsilon_0$, and u_ε, u_δ be such solutions. If $\hat{u}$ is either $u_\varepsilon - u_\delta$ or $(\delta + u_\delta) - (\varepsilon + u_\varepsilon)$ then $\hat{u} \geq 0$ on $\partial\Omega$ and, since g is nonincreasing, $L\hat{u} \geq 0$ on $A = \{x \in \Omega \mid \hat{u}(x) < 0\}$. The maximum principle therefore asserts that $\hat{u}$ attains its minimum over $\bar{A}$ on ∂A. But $\hat{u} = 0$ on ∂A. Since $\hat{u} < 0$ on A, A must be empty. Thus $\hat{u} \geq 0$ and (1. 4) is established.

The next lemma, which follows from [3, Thm.4 of Chapter 5] and the Sobolev embedding theorem, will also be used in later proofs. $W^{k,q}(\Omega)$ denotes the usual Sobolev space in what follows.

Lemma 1.5. Let $\mathscr{D}_0$, $\mathscr{D}$ be bounded open domains in $\mathbb{R}^n$ with $\overline{\mathscr{D}}_0 \subset \mathscr{D}$. Suppose $\mathfrak{L}$ is a second order uniformly elliptic operator with coefficients continuous in $\overline{\mathscr{D}}$ and $q > n$. Then there is a constant K such that

$$(1.6) \qquad \|w\|_{W^{2,q}(\mathscr{D}_0)} \leq K(\|\mathfrak{L}w\|_{L^q(\mathscr{D})} + \|w\|_{L^q(\mathscr{D})})$$

for all $w \in W^{2,q}(\mathscr{D})$. The constant K depends on n, q, the diameter of $\mathscr{D}$, the distance from $\mathscr{D}_0$ to $\partial\mathscr{D}$, the ellipticity constant of $\mathfrak{L}$, and bounds for the coefficients of $\mathfrak{L}$ (in $L^\infty(\mathscr{D})$) and the moduli of continuity of the coefficients.

Completion of proof of Theorem 1.1. From (1.4), $0 \leq u_\varepsilon - u_\delta \leq \delta - \varepsilon$ for $0 < \varepsilon \leq \delta \leq \varepsilon_0$. Therefore $\lim_{\varepsilon \to 0+} u_\varepsilon = u$ exists uniformly in $\overline{\Omega}$ where $u \geq 0$ in Ω and $u = 0$ on $\partial\Omega$. It remains to see that $u \in C^2(\Omega)$ and $Lu = g(x,u)$ in Ω. Clearly $u \geq u_\varepsilon$ for $\varepsilon \in (0, \varepsilon_0]$ by (1.4), so $u > 0$ in Ω and $g(x, \varepsilon + u_\varepsilon) \to g(x,u)$ uniformly on compact subsets of Ω. Choose open subsets $\mathscr{D}_1$, $\mathscr{D}_2$ of Ω so that $\overline{\mathscr{D}}_2 \subset \mathscr{D}_1 \subset \overline{\mathscr{D}}_1 \subset \Omega$ and $q > n$. By Lemma 1.5 there is a constant $K = K(n, q, \mathscr{D}_1, \mathscr{D}_2, L)$ such that

$$(1.7) \qquad \|u_\varepsilon\|_{W^{2,q}(\mathscr{D}_2)} \leq K(\|Lu_\varepsilon\|_{L^q(\mathscr{D}_1)} + \|u_\varepsilon\|_{L^q(\mathscr{D}_1)})$$

$$= K(\|g(x, \varepsilon + u_\varepsilon)\|_{L^q(\mathscr{D}_1)} + \|u_\varepsilon\|_{L^q(\mathscr{D}_1)}) .$$

In conjunction with the remarks above, (1.7) implies that $\{u_\varepsilon \mid \varepsilon \in (0, \varepsilon_0]\}$

is bounded in $W^{2,q}_{loc}(\Omega)$. Therefore $u_\varepsilon \to u$ weakly in $W^{2,q}_{loc}(\Omega)$. Choose $\alpha \in (0,1)$ and $q > n(1-\alpha)^{-1}$. It follows from the Sobolev embedding theorems that $\{u_\varepsilon\}$ is compact in $C^{1,\alpha}_{loc}(\Omega)$ and $g(x, \varepsilon+u_\varepsilon) \to g(x,u)$ in $C^1(\Omega)$. Thus we have $u \in W^{2,p}_{loc}(\Omega)$ and $Lu = g(x,u) \in C^1(\Omega)$. The L^q regularity theory for L [1] now implies $u \in W^{3,q}_{loc}(\Omega)$ and hence $u \in C^{2,\alpha}_{loc}(\Omega) \subset C^2(\Omega)$.

It remains to prove the uniqueness of u. If u and v are classical solutions of (0.1), (g_2) implies $L(u-v) \geq 0$ on $A = \{x \in \Omega \mid u(x) < v(x)\}$. Since $u - v = 0$ on ∂A it follows that $A = \emptyset$ as in the proof of (1.4) and $u \geq v$. By symmetry we also have $v \geq u$ and the proof is complete.

<u>Remark:</u> Rather than using L^q interior estimates as above we could have used the Schauder interior estimates. These are more convenient for making minimal classical regularity assumptions as in [10]. However, we will employ L^q estimates several times below.

We turn now to the more complex case in which $g(x,r)$ is not monotone in r. The existence theorem for this case will be obtained as a consequence of results concerning the more general nonlinear eigenvalue problem

$$(1.8)\qquad \begin{cases} Lu = \lambda g(x,u) & x \in \Omega\,, \\ u = 0 & x \in \partial\Omega\,. \end{cases}$$

By a (classical) solution of (1.8) we mean a pair $(\lambda, u) \in (0,\infty) \times (C^2(\Omega) \cap C_0(\bar{\Omega}))$ with $u > 0$ in Ω which satisfies (1.8). Here and below

$$C_0(\bar{\Omega}) = \{u \in C(\bar{\Omega}) \mid u = 0 \text{ on } \partial\Omega\} .$$

Theorem 1.9. Let g satisfy (g_1). *Then there is a set* $\mathcal{C}$ *of solutions* of (1.8) satisfying

(i) $\mathcal{C}$ is connected in $\mathbb{R} \times C_0(\bar{\Omega})$,

(ii) $\mathcal{C}$ is unbounded in $\mathbb{R} \times C_0(\bar{\Omega})$,

and

(iii) $(0, 0)$ lies in the closure of $\mathcal{C}$ in $\mathbb{R} \times C_0(\bar{\Omega})$.

Theorem 1.9 will be used to prove:

Corollary 1.10. Let g satisfy (g_1) and

$$(g_3) \quad \begin{cases} \text{There is a constant } A \text{ such that} \\ g(x,r) \leq A \text{ for } r \geq 1 \text{ and } x \in \Omega . \end{cases}$$

Let $\mathcal{C}$ be as in Theorem 1.9. Then $\{\lambda \mid (\lambda, u) \in \mathcal{C}\} = (0, \infty)$. In particular, (0.1) has a solution.

Proof of Theorem 1.9. As in the proof of Theorem 1.1 we begin by introducing the approximate problems:

$$(1.11) \qquad Lu = \lambda g(x, \varepsilon + u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega ,$$

where $\varepsilon > 0$. Solutions (λ, u) of (1.11) *have the obvious meaning.* Let u_ν denote the inward normal derivative of u on $\partial\Omega$ and define

$$P = \{u \in C^{1,\alpha}(\bar{\Omega}) \mid u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega \text{ and } u_\nu > 0 \text{ on } \partial\Omega\}$$

where $\alpha \in (0,1)$. Let $0 < \varepsilon \leq \varepsilon_0$ imply $g(x, \varepsilon) > 0$ in Ω. It follows from Theorem 3.7 of [8] that for $0 < \varepsilon \leq \varepsilon_0$ there is a set $\mathcal{C}_\varepsilon$ of solutions of (1.11) which is a connected and unbounded subset of $(\mathbb{R}^+ \times P) \cup \{(0,0)\}$ (in the topology of $\mathbb{R} \times C^{1,\alpha}(\bar{\Omega})$) and contains $(0,0)$. It is easy to see that the topology induced by $\mathbb{R} \times C_0(\bar{\Omega})$ on the set of nonnegative solutions of (1.11) coincides with that induced by $\mathbb{R} \times C^{1,\alpha}(\bar{\Omega})$. Hence $\mathcal{C}_\varepsilon$ is connected and unbounded in $\mathbb{R} \times C_0(\bar{\Omega})$.

The existence of the desired set $\mathcal{C}$ will be obtained by studying $\mathcal{C}_\varepsilon$ as $\varepsilon \to 0+$. Let $\mathcal{O}$ be a bounded open neighborhood of $(0,0)$ in $\mathbb{R} \times C_0(\bar{\Omega})$. The next step in the proof is to find a solution $(\lambda, u) \in \partial\mathcal{O}$ of (1.8). By the properties of $\mathcal{C}_\varepsilon$ there is a solution $(\lambda_\varepsilon, u_\varepsilon) \in \partial\mathcal{O} \cap ((0,\infty) \times P)$ of (1.11) for $0 < \varepsilon \leq \varepsilon_0$. We will show that if $\varepsilon_m \to 0+$ and $\lambda_{\varepsilon_m} \to \lambda$, then $\lambda > 0$, a subsequence of $\{u_{\varepsilon_m}\}$ converges to a function $u \in C^2(\Omega) \cap C_0(\bar{\Omega})$ and (λ, u) has the desired properties. To do this requires upper and lower bounds for u_ε and these are given by the next lemma.

Let $\varphi \in C^{2,\alpha}(\bar{\Omega})$ be defined by

$$L\varphi = 1 \text{ on } \Omega \;;\; \varphi = 0 \text{ on } \partial\Omega . \tag{1.12}$$

<u>Lemma 1.13</u>. Let $M > 0$ and $(\lambda_\varepsilon, u_\varepsilon) \in (0,\infty) \times P$ be a solution of (1.11) satisfying $\lambda_\varepsilon \leq M$ and $u_\varepsilon \leq M$ in $\bar{\Omega}$. There is a number $\bar{\varepsilon} > 0$ and a pair of functions $\bar{\gamma}(M) > 0$, $\bar{K}(\beta, M) > 0$ such that if φ is given by (1.12) and $0 < \varepsilon \leq \bar{\varepsilon}$ then

$$\lambda_\varepsilon \bar{\gamma}(M)\, \varphi(x) \leq u_\varepsilon(x) \leq \beta + \lambda_\varepsilon \bar{K}(\beta, M)\, \varphi(x), \quad x \in \Omega , \tag{1.14}$$

for $\beta \in (0, M]$.

We first use Lemma 1.13 to complete the proof of Theorem 1.9 and then prove the lemma. Since $\mathcal{O}$ is bounded, there is an $M > 0$ such that $(\lambda, u) \in \mathcal{O}$ implies $\lambda, u < M$. Let $(\lambda_\varepsilon, u_\varepsilon) \in \partial\mathcal{O} \cap (0,\infty) \times P$ be solutions of (1.11) as above, $\varepsilon_m \to 0+$ and $\lambda_{\varepsilon_m} \to \lambda$. If $\lambda = 0$ we deduce from (1.14) that

$$0 \le \limsup_{m\to\infty} u_{\varepsilon_m}(x) \le \beta \quad \text{for } \beta \in (0,M]$$

and hence that $u_{\varepsilon_m} \to 0$ in $C_0(\bar{\Omega})$. Thus $(\lambda_{\varepsilon_m}, u_{\varepsilon_m}) \to (0,0)$ in $\mathbb{R} \times C_0(\bar{\Omega})$. Since $(\lambda_{\varepsilon_m}, u_{\varepsilon_m})$ lies in the boundary of the open neighborhood $\mathcal{O}$ of $(0,0)$ this is impossible. Therefore $\lambda > 0$.

From (1.14) and $\lambda > 0$ we see that u_{ε_m} is bounded from below by a function which is positive in Ω and from above by a constant. Arguing as in the proof of Theorem 1.1 we deduce that $\{u_{\varepsilon_m}\}$ is bounded in $W^{2,q}_{loc}(\Omega)$ for $q > n$ and can therefore assume $\{u_{\varepsilon_m}\}$ converges weakly in $W^{2,q}_{loc}(\Omega)$ and strongly in $C^{1,\alpha}_{loc}(\Omega)$ if $q > n/(1-\alpha)$ to a function $u \in C^{2,\alpha}_{loc}(\Omega)$, such that $Lu = \lambda g(x,u)$ in Ω. Passing to the limit in (1.14) we find

$$\lambda \bar{\gamma}(M)\,\varphi(x) \le u(x) \le \beta + \lambda \bar{K}(\beta, M)\,\varphi(x), \quad x \in \bar{\Omega}, \tag{1.15}$$

for $\beta \in (0, M]$. Thus $\limsup_{x\to\partial\Omega} u(x) \le \beta$ for each such β and this implies $\lim_{x\to\partial\Omega} u(x) = 0$. Hence $u \in C_0(\bar{\Omega})$. In a similar way we see that $u_{\varepsilon_m}(x) \to 0$ as $x \to \partial\Omega$ uniformly in m. From this and $u_{\varepsilon_m} \to u$ in $C(\Omega)$ it follows that $(\lambda_{\varepsilon_m}, u_{\varepsilon_m}) \to (\lambda, u)$ in $\mathbb{R} \times C_0(\bar{\Omega})$ and hence $(\lambda, u) \in \partial\mathcal{O}$.

At this point we have shown that if $\mathcal{O}$ is a bounded neighborhood of $(0,0)$ in $\mathbb{R} \times C_0(\Omega)$ then there is a solution $(\lambda, u) \in \partial\mathcal{O}$ of (1.8) which moreover satisfies (1.15). Let $\mathfrak{F}$ denote the set of solutions of (1.11) which satisfy (1.15) with $M = \max\{\lambda, \|u\|_{C(\bar{\Omega})}\}$ (as do the ones we have obtained). Since $\mathfrak{F} \cap \partial\mathcal{O} \neq \emptyset$ for $\mathcal{O}$ as above, if closed and bounded (in $\mathbb{R} \times C_0(\bar{\Omega})$) subsets of $\mathfrak{F}$ are compact the existence of $\mathcal{C} \subset \mathfrak{F}$ possessing the desired properties follows from a standard argument of point set topology [12]. Let $\{(\lambda_m, u_m)\}$ be a bounded sequence in $\mathfrak{F}$. Without loss of generality we can assume $\lambda_m \to \lambda \geq 0$. If $\lambda = 0$, $u_m \to 0$ via (1.15). If $\lambda > 0$, one shows as above that a subsequence of $\{u_m\}$ converges in $C^2(\Omega) \cap C_0(\bar{\Omega})$ to a function u such that $(\lambda, u) \in \mathfrak{F}$. Thus to complete the proof of Theorem 1.9 it remains to prove Lemma 1.13.

<u>Proof of Lemma 1.13</u>. Set

$$\bar{K}(\beta, M) = \max\{g(x,r) \mid x \in \bar{\Omega},\ \beta \leq r \leq 1 + M\}\ . \tag{1.16}$$

Let $(\lambda_\varepsilon, u_\varepsilon)$ be as in the lemma, $0 < \varepsilon \leq 1$ and $\beta \in (0, M]$. Set $A_\beta = \{x \in \Omega \mid u_\varepsilon(x) > \beta\}$. Then, using (1.12), (1.16),

$$L(\beta + \lambda_\varepsilon \bar{K}(\beta, M)\varphi - u_\varepsilon) = c\beta + \lambda_\varepsilon(\bar{K}(\beta, M) - g(x, \varepsilon + u_\varepsilon)) \geq 0, \quad x \in A_\beta$$

$$\beta + \lambda_\varepsilon \bar{K}(\beta, M)\varphi - u_\varepsilon = \lambda_\varepsilon \bar{K}(\beta, M)\varphi \geq 0, \qquad x \in \partial A_\beta\ .$$

Thus $u_\varepsilon \leq \beta + \lambda_\varepsilon \bar{K}(\beta, M)\varphi$ on A_β *by the maximum principle and the* right-hand inequality of (1.13) is established.

To obtain the left-hand inequality, choose $R > 0$ so that $g(x, r) > 1$ if $0 < r < R$, which is possible by (g_1). Define $\bar{\gamma} = \min(1, R/2M\|\varphi\|_{C(\bar{\Omega})})$. Then for $\varepsilon \in [0, R/2]$, $\gamma \in (0, \bar{\gamma}]$ and $\lambda \in (0, M]$

$$(1.17) \qquad L\lambda\gamma\varphi < \lambda g(x, \varepsilon + \lambda\gamma\varphi) \text{ in } \Omega .$$

From this we will deduce that $\lambda_\varepsilon \bar{\gamma}\varphi \leq u_\varepsilon$. Indeed, define γ^* by $u_\varepsilon - \gamma\lambda_\varepsilon \varphi \in P$ for $0 \leq \gamma < \gamma^*$ and $u_\varepsilon - \lambda_\varepsilon \gamma^* \varphi \notin P$. It suffices to show $\gamma^* > \bar{\gamma}$. If $\gamma^* \leq \bar{\gamma}$, $w = u_\varepsilon - \gamma^* \lambda_\varepsilon \varphi \geq 0$ in Ω and, by (1.17), for $C > 0$

$$(1.18) \qquad Lw + Cw > Cw + \lambda_\varepsilon [g(x, \varepsilon + u_\varepsilon) - g(x, \varepsilon + \lambda_\varepsilon \gamma^* \varphi)] .$$

But $g(x, \varepsilon + u_\varepsilon) - g(x, \varepsilon + \gamma^* \lambda_\varepsilon \varphi) \geq C_0 w$ where C_0 is a lower bound on $\frac{\partial g}{\partial r}(x, r)$ for $\varepsilon \leq r \leq \varepsilon + \|u_\varepsilon\|_{C(\bar{\Omega})} + \lambda_\varepsilon \gamma^* \|\varphi\|_{C(\bar{\Omega})}$. Choosing C so that $C + \lambda_\varepsilon C_0 > 0$, $Lw + Cw > 0$ by (1.18) and $w \in P$ by the maximum principle. This contradicts the definition of γ^* and so $\gamma^* > \bar{\gamma}$. Hence $\bar{K}$ above, $\bar{\gamma} = \min(1, R/2M\|\varphi\|_{C(\bar{\Omega})})$, $\bar{\varepsilon} = R/2$ have the desired properties and the proof is complete.

<u>Proof of Corollary 1.10.</u> By the assumptions of the Corollary, (1.15) and (1.16), $(\lambda, u) \in \mathcal{C}$ implies (choosing $\beta = 1$)

$$(1.19) \qquad u(x) \leq 1 + \lambda A\varphi(x), \quad x \in \Omega .$$

Since $\mathcal{C}$ is connected and $(0, 0) \in \bar{\mathcal{C}}$, the projection $(\lambda, u) \in \mathcal{C} \to \lambda$ of $\mathcal{C}$ into $\mathbb{R}$ has as its range an interval $(0, \Lambda)$ or $(0, \Lambda]$. If $\Lambda < \infty$, (1.19) implies that $\mathcal{C}$ is bounded in $\mathbb{R} \times C_0(\bar{\Omega})$. Since $\mathcal{C}$ is unbounded,

$\Lambda = \infty$ and the proof is complete.

We conclude this section with an extension of Theorem 1.9 to the case in which the coefficients of L and g are merely continuous. More precisely, we assume that

(L) $\quad \begin{cases} L \text{ is uniformly elliptic with continuous coefficients} \\ \text{in } \bar{\Omega} \text{ and } c(x) \geq 0 \text{ in } \Omega \text{ ,} \end{cases}$

and, in addition to (g_1),

(g_4)
$$g \in C(\bar{\Omega} \times (0, \infty)) \ .$$

One cannot expect classical solutions of

$$\text{(1.20)} \qquad Lu = \lambda g(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

in this case. We say (λ, u) is a <u>generalized solution</u> of (1.20) provided that there is a $q > n$ for which $(\lambda, u) \in (0, \infty) \cap (C_0(\bar{\Omega}) \cap W^{2,q}_{loc}(\Omega))$ and (1.20) is satisfied a.e.

<u>Theorem 1.21.</u> Let (L), (g_1) and (g_4) be satisfied. Then (1.20) has a set $\mathcal{C}$ of solutions satisfying the assertions of Theorem 1.9.

<u>Proof of Theorem 1.21.</u> The proof is similar to that of Theorem 1.9 so the exposition will be abbreviated. For approximate problems we consider

$$\text{(1.22)} \qquad L_m u = \lambda g_m(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

where

$$L_m = - \sum_{i,j=1}^{n} a_{ijm}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_{im}(x) \frac{\partial}{\partial x_i} + c_m(x)$$

is a sequence of elliptic operators with $c_m \geq 0$ whose coefficients are smooth and converge as $m \to \infty$ to the coefficients of L uniformly in $\bar{\Omega}$. The functions g_m are chosen to be smooth in $\bar{\Omega} \times [0, \infty)$, to satisfy $g_m(x, r) \geq 1$ for $0 < r < R$ for some R independent of m and $g_m \to g$ uniformly on compact subsets of $\bar{\Omega} \times (0, \infty)$ as $m \to \infty$. (For example, let g_m be a smooth approximation in the maximum norm of $g(x, \frac{1}{m} + r)$ on $\bar{\Omega} \times [0, m]$.)

The program now runs much as before. Let $\mathfrak{S}$ be as in the proof of Theorem 1.9. As earlier, there is a solution $(\lambda_m, u_m) \in \partial\mathfrak{S} \cap ((0, \infty) \times P)$ of (1.22). To obtain a solution of (1.20) in $\partial\mathfrak{S}$ from (λ_m, u_m) as $m \to \infty$ requires a version of (1.15). From the proof of (1.15) we have

$$\lambda_m \bar{\gamma}_m(M)\, \varphi_m(x) \leq u_m(x) \leq \beta + \lambda_m \bar{K}_m(\beta, M)\, \varphi_m(x) \quad \text{in } \Omega \tag{1.23}$$

provided $\lambda_m, u_m \leq M$ and $\beta \in (0, M]$ where

$$\begin{cases} \bar{K}_m(\beta, M) = \max\{g_m(x, r) : x \in \bar{\Omega},\ \beta \leq r \leq 1 + M\} \\ \\ L_m \varphi_m = 1 \text{ in } \Omega, \quad \varphi_m = 0 \text{ on } \partial\Omega \,. \end{cases} \tag{1.24}$$

and $g_m(x, r) > 1$ if $0 < r < R$. By the results of Bony [4] $L: W_0^{2,q}(\Omega) \to L^q(\Omega)$ is an isomorphism for $q > n$. Let $\mathcal{B}(X, Y)$ be the space of bounded linear operators from X to Y. Since $L_m \to L$ in $\mathcal{B}(W_0^{2,q}(\Omega), L^q(\Omega))$ as $m \to \infty$,

$$L_m^{-1} \to L^{-1} \text{ in } \mathcal{B}(L^q(\Omega),\ W_0^{2,q}(\Omega))$$

and a fortiori $L_m^{-1} \to L^{-1}$ in $\mathcal{B}(L^q(\Omega), C_0(\bar{\Omega}))$ if $q > n$. In particular,

$\varphi_m \to \varphi$ in $C_0(\overline{\Omega})$ where φ is the unique solution of $L\varphi = 1$ in $W_0^{2,q}(\varphi)$. Let $\overline{\gamma}$, $\overline{K}$ be the limits as $m \to \infty$ of $\overline{\gamma}_m, \overline{K}_m$ defined in (1.24). Note also that Bony's results imply $\varphi > 0$ in Ω.

It is now a simple matter to complete the proof as before. Choosing appropriate subsequences if necessary $\lambda_m \to \lambda > 0$, $u_m \to u$ in $W_{loc}^{2,q}(\Omega) \cap C_0(\Omega)$, and $(\lambda, u) \in \partial\mathcal{O}$ is a solution of (1.20). Moreover,

$$\lambda\,\overline{\gamma}\,(M)\,\varphi(x) \leq u(x) \leq \beta + \lambda\overline{K}(\beta, M)\,\varphi(x) \tag{1.23}$$

for $\beta \in (0, M]$, $\lambda, u \leq M$. It then suffices to show any closed bounded subset of generalized solutions of (1.20) satisfying (1.23) is compact in $(0,\infty) \times C_0(\Omega) \cup \{0, 0\}$. But this follows as before as well, and the proof is complete.

Remark 1.25. Corollary (1.10) remains true if $\mathcal{C}$ is the set obtained in Theorem 1.21.

Remark 1.26. It is clear that the existence assertion of Theorem 1.1 follows from Corollary 1.10 and that Theorem 1.9 may be deduced from Theorem 1.21 and standard regularity theorems. However our presentation was organized so as to minimize duplication of arguments and, hopefully, to minimize the readers discomfort.

Section 2. Boundary Behaviour and Regularity.

This section is devoted to the study of the behaviour of solutions of (0.1) near the boundary. In particular, if g is independent of x and satisfies (g_2) the precise rate at which $u(x) \to 0$ as $x \to \partial\Omega$ is determined and an estimate for $|\text{grad } u(x)|$ near $\partial\Omega$ is obtained. This information is then used to obtain a modulus of continuity (sometimes precise) for u in $\bar{\Omega}$.

We assume below that g is independent of x and that (g_1) holds. The estimates for u and $|\text{grad } u|$ will be in terms of a solution of the one-dimensional problem $-p'' = g(p)$. To be precise, let $a > 0$ and $p \in C^2((0,a]) \cap C([0,a])$ satisfy

$$(2.1) \quad \begin{cases} \text{(i)} & -p''(s) = g(p(s)) \text{ for } 0 < s \leq a , \\ \text{(ii)} & p(0) = 0 , \\ \text{(iii)} & p(s) > 0 \text{ and } g(p(s)) > 0 \text{ for } 0 < s \leq a . \end{cases}$$

Our main estimates for u and $|\text{grad } u|$ are

__Theorem 2.2.__ Let $g = g(r)$ be continuous and satisfy (g_1), (g_2). Let L satisfy (L), and $u \in C_0(\Omega) \cap W^{2,q}_{loc}(\Omega)$ be a solution of (0,1) where $q > n$. If p satisfies (2.1), then there are constants $\lambda, \Lambda > 0$ such that

$$(2.3) \quad \lambda p(d(x)) \leq u(x) \leq \Lambda p(d(x)) \text{ on } \Omega_a = \{x \in \Omega \mid d(x) \leq a\}$$

where

$$(2.4) \quad d(x) = \text{distance } (x, \partial\Omega) .$$

__Theorem 2.5.__ Under the hypotheses of Theorem 2.2 there are constants $M, m, a > 0$ such that

$$(2.6) \quad |\text{grad } u(x)| \leq M[d(x)\, g(mp(d(x))) + \frac{p(d(x))}{d(x)}]$$

on Ω_a.

As an illustration of these results, consider the special case $g(r) = r^{-\alpha}$. Trying for a solution of $-p'' = p^{-\alpha}$ of the form $p(s) = bs^{\beta}$ leads to the equations $-b\beta(\beta-1) = b^{-\alpha}$ and $\beta - 2 = -\alpha\beta$. If $\alpha > 1$, then $\beta = 2(1+\alpha)^{-1}$ and the equation for b has a unique solution. Thus Theorem 2.2 implies that $u(x)$ is bounded from above and from below by a multiple of $d(x)^{2/(1+\alpha)}$. Theorem 2.5 further tells us that $|\operatorname{grad} u(x)|$ is bounded above by a multiple of $d(x)^{(1-\alpha)/(1+\alpha)}$. Together these estimates imply that for $\gamma > 0$

$$|\operatorname{grad}(u(x)^{\gamma})| \le \gamma |u(x)|^{\gamma-1} |\operatorname{grad} u(x)| \le K_2\, d(x)^{\mu}$$

where $\mu = [2(\gamma-1) + (1-\alpha)]/(1+\alpha)$. Choosing $\gamma = (1+\alpha)/2$ we have $\mu = 0$ and $|\operatorname{grad}(u(x)^{(1+\alpha)/2})|$ is bounded. Thus $u(x)^{(1+\alpha)/2}$ is Lipschitz continuous on $\bar{\Omega}$. Since $v(x)^{\theta}$ is Hölder continuous with exponent θ whenever $\theta \in (0,1)$ and $v(x)$ is Lipschitz continuous, $u(x) = (u(x)^{(1+\alpha)/2})^{2/(1+\alpha)}$ is Hölder continuous with exponent $2/(1+\alpha)$ in $\bar{\Omega}$. It is interesting that this is precisely the modulus of continuity of solutions of (2.1).

These remarks show how Theorems 2.2 and 2.5 can be employed to obtain a modulus of continuity for u. Namely, if we can find a monotone function f so that $\operatorname{grad} f(u) = f'(u) \operatorname{grad} u$ is bounded, then $f(u)$ is Lipschitz continuous and the continuity of $u = f^{-1}(f(u))$ is essentially that of f^{-1}.

Our program in the remainder of this section is as follows: First a detailed study of the problem $-p'' = g(p)$, $p(0) = 0$ is made. This will clarify the freedom of choice of p in Theorems 2.2 and 2.5 and provide other information used later as well. Then Theorems 2.2 and 2.5 are proved. Next we remark on the case in which g depends on x and is not monotone. Lastly, a further study is made of the modulus of continuity of u by the method described above.

<u>Lemma 2.7.</u> Let $g: (0, \infty) \to \mathbb{R}$ be continuous and $g(r) \to +\infty$ as $r \to 0+$. Assume $g(r) > 0$ for $0 < r \leq b$ and set

$$F(s) = \int_s^b g(\tau)d\tau \quad \text{for } s > 0 \tag{2.8}$$

and

$$h(s) = \int_0^s \frac{1}{\sqrt{2F(\tau)}} d\tau \quad \text{for } 0 < s \leq b . \tag{2.9}$$

If p satisfies (2.1), then

$$\ell \equiv \lim_{s \to 0+} \frac{h^{-1}(s)}{p(s)}$$

exists and is positive. If $F(0+) = \infty$, then $\ell = 1$.

<u>Proof of Lemma 2.7.</u> From $-p'' = g(p)$ we deduce that

$$p'(s)^2 = 2F(p(s)) + \bar{C} \tag{2.10}$$

for some constant $\bar{C}$. Since $-p'' = g(p) > 0$, p is concave near $s = 0$ and, although possibly infinite, $p'(0+)$ exists. Moreover, $p(s) > 0$ for $0 < s < a$ and therefore $p'(0+) > 0$. Thus $0 < p'(0+) \leq \infty$ and $p'(s) > 0$ near $s = 0$. Now (2.10) implies $s = H(p(s))$ for small s where

$$H(s) = \int_0^s \frac{1}{\sqrt{2F(\tau)+\bar{C}}}\, d\tau .$$

Therefore $p(s) = H^{-1}(s)$. (If $\bar{C} = 0$, h as given by (2.9) coincides with H. Moreover, h^{-1} has the properties required of p.) It is clear from (2.10) that $p'(0+) < \infty$ *if and only if* $F(0+) < \infty$. *Hence* when $F(0+) < \infty$ l'Hospital's rule implies

$$0 < \lim_{s \to 0+} \frac{h^{-1}(s)}{p(s)} = \frac{\sqrt{2F(0+)}}{\sqrt{2F(0+)+\bar{C}}} = \ell < \infty .$$

If $F(0+) = \infty$, the definitions of h, H imply that for $\varepsilon \in (0,1)$ there is an $r_\varepsilon > 0$ for which

(2.11) $$(1-\varepsilon)\, h(r) \le H(r) \le (1+\varepsilon)\, h(r), \qquad 0 \le r < r_\varepsilon .$$

Choose $r = p(s)$ in (2.11) and recall that $p = H^{-1}$ to deduce that

(2.12) $$h^{-1}\left(\frac{s}{1+\varepsilon}\right) \le p(s) \le h^{-1}\left(\frac{s}{1-\varepsilon}\right) \text{ if } 0 < p(s) \le r_\varepsilon .$$

Since h^{-1} is concave near 0 and $h^{-1}(0) = 0$, $h^{-1}(\alpha s) \ge \alpha h^{-1}(s)$ and $h^{-1}(s) = h^{-1}(\frac{1}{\gamma}\gamma s) \ge \frac{1}{\gamma} h^{-1}(\gamma s)$ for $\alpha \in [0,1]$, $\gamma \ge 1$ and $s, \gamma s > 0$ sufficiently small. Thus (2.12) implies that

(2.13) $$\frac{1}{1+\varepsilon} h^{-1}(s) \le p(s) \le \frac{1}{1-\varepsilon} h^{-1}(s)$$

for s near 0. Since $\varepsilon \in (0,1)$ is arbitrary, (2.13) implies $\ell = 1$ in this case.

<u>Proof of Theorem 2.2.</u> Since $u(x)$ and $p(d(x))$ are positive on Ω_a,

it suffices to show that we can find λ, Λ, $0 < a' \leq a$ for which (2.3) holds on $\Omega_{a'}$. Then (2.3) will hold on Ω_a with some other choice of λ, Λ.

Set

$$\hat{L} = L - c = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i}$$

and define $\varphi \in W_0^{2,q}(\Omega)$ (see [4]) by $\hat{L}\varphi = 1$. By the maximum principle of Bony ([4]) $\varphi > 0$ in Ω and, as a simple comparison argument shows the interior normal derivative of φ on $\partial\Omega$ is positive. Thus $\varphi(x)$ is bounded above and below on $\bar{\Omega}$ by positive multiples of $d(x)$. It is therefore enough to show that u may be bounded above and below near $\partial\Omega$ by positive multiples of $p(\varphi)$ since

(2.14) $\beta p(s) \leq p(\beta s)$, $p(\gamma s) \leq \gamma p(s)$ for $0 \leq \beta \leq 1$, $\gamma \geq 1$ and $0 \leq \gamma s \leq a$.

The inequalities (2.14) follow from the concavity of p as in the proof of Lemma 2.7.

Let $\mathcal{D}_r = \{x \in \Omega \mid \varphi(x) < r\}$ where $r > 0$ is so small that $\mathcal{D}_r \subset \Omega_a$ and $p'(s) > 0$ for $0 \leq s \leq r$. Next observe that if $\bar{v} \in W^{2,q}(\mathcal{D}_r)$, $L\bar{v} \geq g(\bar{v})$ in $\mathcal{D}_r$ and $\bar{v} \geq u$ on $\partial\mathcal{D}_r$ then $\bar{v} \geq u$ in $\mathcal{D}_r$ by the maximum principle of Bony (see the uniqueness proof of Theorem 1.1). Similarly, if $L\underline{v} \leq g(\underline{v})$ and $\underline{v} \leq u$ on $\partial\mathcal{D}_r$, then $\underline{v} \leq u$ in $\mathcal{D}_r$.

We will show that we may choose $\bar{v} = \Lambda p(\varphi)$, $\underline{v} = \lambda p(\varphi)$ if Λ is sufficiently large and $0 < \lambda$ is sufficiently small. By direct computation

$$(2.15)\quad Lp(\varphi) = p'(\varphi)\,\hat{L}\varphi + cp(\varphi) - p''(\varphi)\sum_{i,j=1}^{n} a_{ij}(x)\,\varphi_{x_i}\varphi_{x_j}$$
$$= p'(\varphi) + cp(\varphi) + g(p(\varphi))\sum_{i,j=1}^{n} a_{ij}(x)\,\varphi_{x_i}\varphi_{x_j}\,.$$

Observe first that for r sufficiently small there is a $\delta > 0$ such that $|\mathrm{grad}\,\varphi(x)| \geq \delta$ for $x \in \mathcal{D}_r$. Since $p'(\varphi) > 0$ in $\mathcal{D}_r$ and $c \geq 0$, to prove

$$L\Lambda p(\varphi) \geq g(\Lambda\, p(\varphi)) \quad \text{in } \mathcal{D}_r$$

it suffices to choose Λ so that

$$(2.16)\qquad \Lambda g(p(\varphi))\sum_{i,j=1}^{n} a_{ij}(x)\,\varphi_{x_i}\varphi_{x_j} \geq g(\Lambda\, p(\varphi)) \quad \text{in } \mathcal{D}_r\,.$$

By the uniform ellipticity of L, there is a $\mu > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\varphi_{x_i}\varphi_{x_j} \geq \mu\,|\mathrm{grad}\,\varphi|^2$$

on Ω. Using the monotonicity of g, (2.16) holds provided that $\Lambda\delta^2\mu > 1$ and $\Lambda > 1$. Finally, $\Lambda p(\varphi) \geq u$ on $\partial\mathcal{D}_r$ also holds when Λ is large enough. Thus then we have established an upper bound of the form (2.4).

To show that $\lambda p(\varphi)$ has the desired properties, observe first that by (2.15) and (g_1) we have

$$(2.17)\qquad L(\lambda p(\varphi)) \leq g(\lambda p(\varphi)) \quad \text{on } \mathcal{D}_r$$

for sufficiently small $\lambda > 0$ provided that, e.g., (2.18) $$2\lambda p'(\varphi) \leq g(\lambda p(\varphi))$$

on $\mathcal{D}_r$. By (2.10), (2.8) and the monotonicity of g

$$p'(\varphi) \leq \sqrt{2(b-p(\varphi))\,g(p(\varphi))+\overline{c}} \leq c_1(\sqrt{g(p(\varphi))+1} \leq c_2\sqrt{g(\lambda p(\varphi))}$$

for $\lambda > 0$ small enough and some constants c_1, c_2. It is thus clear that λ can be chosen to satisfy (2.18) and also $\lambda p(\varphi) \leq u$ on $\partial\Omega_r$. This completes the proof.

Proof of Theorem 2.5. We are indebted to L. Nirenberg for the basic idea of the proof. The first step is the following lemma.

Lemma 2.19. There is a constant $K_1 > 0$ such that if $r \in (0,1]$, $B_{2r}(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < 2r\} \subseteq \Omega$ and $v \in W^{2,q}(B_{2r}(x_0))$ where $q > n$ then

(2.20) $$|\text{grad}\, v(x)| \leq K_1(r\|Lv\|_{L^\infty(B_{2r}(x_0))} + \frac{1}{r}\|v\|_{C(B_{2r}(x_0))})$$

for $x \in B_r(x_0)$. ($\|Lv\|_{L^\infty(B_{2r}(x_0))} = \infty$ is allowed.)

The lemma is proved after first being used to establish Theorem 2.5. Let $x \in \Omega_a$ and set $r = d(x)/3$, $v = u$ (so $Lv = Lu = g(u)$) and $x = x_0$ in (2.20). Note that if $z \in B_{2r}(x_0)$ then

(2.21) $$d(x)/3 \leq d(z) \leq 5d(x)/3 .$$

Thus if $A = \{z \in \Omega \mid d(x) \leq 3d(z) \leq 5d(x)\}$, then

(2.22) $\quad |\operatorname{grad} u(x)| \le K_2(d(x)\|g(u)\|_{L^\infty(A)} + \frac{1}{d(x)}\|u\|_{L^\infty(A)})$

where $K_2 = 3K_1$. Now Theorem 2.2 together with the monotonicity of g, (2.14) and (2.22) yield the assertions of the theorem.

Proof of Lemma 2.19. Let $x_0 \in \Omega$, and $0 < 2r < d(x_0)$. Changing variables according to $x_0 + ry = x$, we define

$$L_{rx_0} = \frac{1}{r^2}\sum_{i,j=1}^{n} a_{ij}(x_0+ry)\frac{\partial^2}{\partial y_i \partial y_j} + \frac{1}{r}\sum_{i=1}^{n} b_i(x_0+ry)\frac{\partial}{\partial x_i} + c(x_0+ry)$$

and $v_{rx_0}(y) = v(x_0+ry)$ for $|y| \le 2$ and $v \in W^{2,q}(B_{2r}(x_0))$. Then

$$(L_{rx_0} v_{rx_0})(y) = (Lv)(x) \quad \text{for} \quad |y| \le 2 .$$

The operators $r^2 L_{rx_0}$ in $\{|y| \le 2\}$ have uniformly bounded coefficients, a uniform ellipticity constant and the coefficients have a uniform modulus of continuity for $x_0 \in \Omega$, $0 < 2r < d(x_0)$. Thus by the Sobolev embedding Theorem and Lemma 1.5

$$C\left|\frac{\partial}{\partial y_i} v_{rx_0}(y)\right| \le K(r^2\|L_{rx_0} v_{rx_0}\|_{L^q(|y|\le 2)} + \|v_{rx_0}\|_{L^q(|y|\le 2)})$$

for $|y| \le 1$. Estimating the L^q norms by the L^∞ norms and using $\frac{\partial}{\partial y_i} v_{rx_0}(y) = r\frac{\partial}{\partial x_i} v(x)$ above then supplies the desired estimate.

Remark 2.23. If $g(x,r)$ depends on x and is not monotone in r suppose there are continuous $\underline{g}$, $\overline{g}$ such that

(2.24) $\quad \underline{g}(r) \le \inf_{0<s\le r} \min_{x\in\Omega} g(x,s)$ and $\overline{g}(r) \ge \sup_{s\ge r} \max_{x\in\overline{\Omega}} g(x,s)$.

(If $g(x,r)$ is monotone in r, we can drop the inf and sup and define $\underline{g}, \overline{g}$ by equality in (2.24). For the general case there may not exist such a continuous $\overline{g}$).

Then solutions of (0.1) are bounded above by solutions of $L\bar{u} = \bar{g}(u)$, $\bar{u} = 0$ on $\partial\Omega$ and below by solutions of $L\underline{u} = \underline{g}(\underline{u})$, $\underline{u} = 0$ on $\partial\Omega$. Let $\bar{p}$, $\underline{p}$ satisfy (2.1) with g replaced by $\bar{g}$ and $\underline{g}$ respectively. Then the above arguments establish estimates of the form $\lambda\underline{p}(d) \leq u \leq \Lambda\bar{p}(d)$ and $|\text{grad } u| \leq K(d\,\bar{g}(M\underline{p}(d)) + \bar{p}(d)/d)$ in a neighborhood of $\partial\Omega$ in $\bar{\Omega}$.

We now return to the question of obtaining an estimate for the modulus of continuity of u. The case $g(r) = r^{-\alpha}$, $\alpha > 1$ was discussed earlier. The cases $0 < \alpha < 1$ are covered by the next result.

Theorem 2.25. Under the hypotheses of Theorem 2.2, the following are equivalent:

(i) u is Lipschitz continuous in $\bar{\Omega}$

(ii) $F(0+) < \infty$ where F is given by (2.8).

Proof of Theorem 2.25. First recall from the proof of Lemma 2.7 that $p'(0+) = \infty$ if and only if $F(0+) = \infty$. Let $\nu(x)$ be the inward pointing unit normal on $\partial\Omega$. By Theorem 2.2, if $x \in \partial\Omega$ and $t > 0$

$$\frac{\lambda p(d(x+t\nu(x)))}{t} \leq \frac{u(x+t\nu(x))}{t} \leq \frac{\Lambda p(d(x+t\nu(x)))}{t}. \tag{2.26}$$

Since $d(x + t\nu(x)) = t + o(t)$, if u is Lipschitz continuous (2.26) implies that $p'(0+) < \infty$. In fact, if $p'(0+) = \infty$ we see from (2.26) that $\liminf_{t \to 0+} u(x+t\nu(x))/t = u_\nu(x) = \infty$ at every $x \in \partial\Omega$. It remains to show that u is Lipschitz continuous when $F(0+) < \infty$. By Theorem 2.5, it suffices to bound $p(s)/s$ and $sg(p(s))$ for s near 0. But $p(s)/s \to p'(0+) < \infty$ as $s \to 0+$, so $p(s)/s$ is bounded. Thus we need only bound

$sg(s)$. Since $F(0+) - F(s) = \int_0^s g(\tau)d\tau \geq sg(s)$ by the monotonicity of g, if $F(0+) < \infty$ then $sg(s) \to 0$ as $s \to 0+$ and the proof is complete.

Next we treat the case $g(r) = r^{-1}$.

<u>Theorem 2.27:</u> If $g(r) = r^{-1}$ and u and p satisfy (0.1) and (2.1) respectively, then $p^{-1}(u)$ is Lipschitz continuous near $\partial\Omega$.

<u>Proof of Theorem 2.27</u>. Using (2.10) and Theorem 2.5 we have

$$(2.28) \qquad |\text{grad } p^{-1}(u)| = \frac{1}{\sqrt{2F(u)+\overline{c}}}\,|\text{grad } u| \leq \frac{K}{\sqrt{2F(u)+\overline{c}}}\left(\frac{d}{p(d)} + \frac{p(d)}{d}\right)$$

for some constant K. Next, choosing $b = 1$,

$$(2.29) \qquad F(u) = \int_u^1 \frac{1}{\tau}\,d\tau = -\log u \geq -\log(\Lambda\, p(d))$$

by Theorem 2.2. The term $d/p(d)$ on the right in (2.28) tends to 0 as $d \to 0$ since $p'(0+) = \infty$. Thus it remains to show that $p(d)/(d\sqrt{-\log(\Lambda p(d)) + \overline{c}})$ is bounded as $d \to 0$ and to do this it suffices to bound $p(d)/d\sqrt{-\log(p(d))}$. Setting $s = p(d)$, this expression can be written $s/(H(s)\sqrt{-\log s})$ where

$$H(s) = \int_0^s \frac{1}{\sqrt{-2\log\tau + \overline{c}}}\,d\tau \geq \int_{s/2}^s \frac{1}{\sqrt{-2\log\tau+\overline{c}}}\,d\tau \geq \frac{s}{2}\,\frac{1}{\sqrt{-2\log(\frac{s}{2}) + \overline{c}}}.$$

Hence $s/(H(s)\sqrt{-\log s})$ is bounded as $s \to 0+$ and the proof is complete.

More generally, if g is independent of x and nonincreasing, we seek an increasing convex function $f \geq 0$ with $f(0) = 0$ such that $f(u)$ is Lipschitz continuous near $\partial\Omega$. By Theorems 2.2 and 2.5,

$$|\text{grad } f(u)| = |f'(u) \text{ grad } u| \leq K f'(\Lambda p(d)) \left[d g(M p(d)) + \frac{p(d)}{d} \right],$$

or, with $p = h^{-1}$, h given by (2.9), $s = \Lambda p(d)$, $d = h(s/\Lambda)$,

$$|\text{grad } f(u)| \leq K f'(s) \left(h\left(\frac{s}{\Lambda}\right) g\left(\frac{M}{\Lambda} s\right) + \frac{s}{h\left(\frac{s}{\Lambda}\right)} \right). \tag{2.30}$$

It therefore suffices to bound the right hand side of (2.28) by a constant. If we choose $f'(s) = h(s/\Lambda)/g(Ms/\Lambda)$, then we can clearly bound (2.28) and $f'(s)$ is increasing as desired. This choice is quite crude and can be improved. For example, if $g(r) = r^{-\alpha}$, $\alpha > 1$, $f(s) \sim s^{(3\alpha+1)/2}$ and we would only conclude $u \in C^{2/(3\alpha+1)}(\bar{\Omega})$. We do not know an optimal continuity result for the general case. Perhaps Theorem 2.5 is itself too crude for this purpose.

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Elliptic boundary value problems of the form $Lu = g(x, u)$ in Ω and $u = 0$ on the boundary of Ω are studied where g is singular in that $g(x, r) \to \infty$ uniformly as $r \to 0+$. Existence of classical and generalized solutions is established and an associated nonlinear eigenvalue problem is treated. A detailed study is made of the behaviour of the solutions and their gradients near the boundary of Ω. This leads to global estimates for the modulus of continuity of solutions.

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